1077. Let $f(x)$ be a continuous function for all real numbers x in the interval $[a,b]$ except for some point c in (a,b). Then $\int_a^b f(x)dx = \lim_{\varepsilon\to 0+} \int_a^{c-\varepsilon} f(x)dx + \lim_{\delta\to 0+} \int_{c+\delta}^b f(x)dx$.

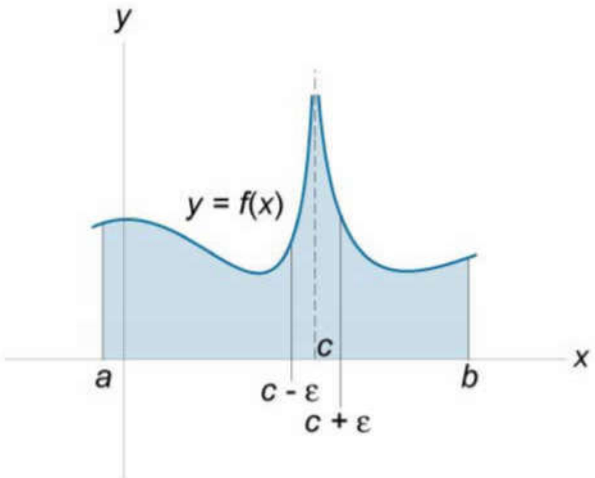

Figure 188.

9.10 Double Integral

Functions of two variables: $f(x,y)$, $f(u,v)$, ...

Double integrals: $\iint_R f(x,y)dxdy$, $\iint_R g(x,y)dxdy$, ...

Riemann sum: $\sum_{i=1}^{m}\sum_{j=1}^{n} f(u_i,v_j)\Delta x_i \Delta y_j$

Small changes: Δx_i, Δy_j

Regions of integration: R, S

Polar coordinates: r, θ

Area: A
Surface area: S
Volume of a solid: V
Mass of a lamina: m
Density: $\rho(x, y)$
First moments: M_x, M_y
Moments of inertia: I_x, I_y, I_0
Charge of a plate: Q
Charge density: $\sigma(x, y)$
Coordinates of center of mass: $\bar{x}$, $\bar{y}$
Average of a function: μ

1078. Definition of Double Integral
The double integral over a rectangle $[a, b]\times[c, d]$ is defined to be

$$\iint\limits_{[a,\,b]\times[c,\,d]} f(x, y)dA = \lim_{\substack{\max \Delta x_i \to 0 \\ \max \Delta y_j \to 0}} \sum_{i=1}^{m}\sum_{j=1}^{n} f(u_i, v_j)\Delta x_i \Delta y_j ,$$

where (u_i, v_j) is some point in the rectangle $(x_{i-1}, x_i)\times(y_{j-1}, y_j)$, and $\Delta x_i = x_i - x_{i-1}$, $\Delta y_j = y_j - y_{j-1}$.

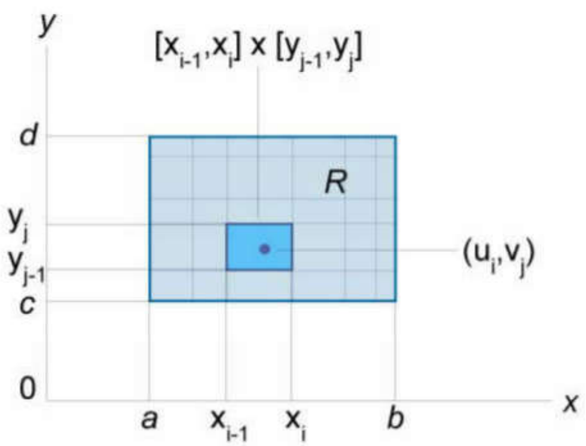

Figure 189.

The double integral over a general region R is

$$\iint_R f(x,y)\,dA = \iint_{[a,\,b]\times[c,\,d]} g(x,y)\,dA,$$

where rectangle $[a, b]\times[c, d]$ contains R,

$g(x,y)=f(x,y)$ if $f(x,y)$ is in R and $g(x,y)=0$ otherwise.

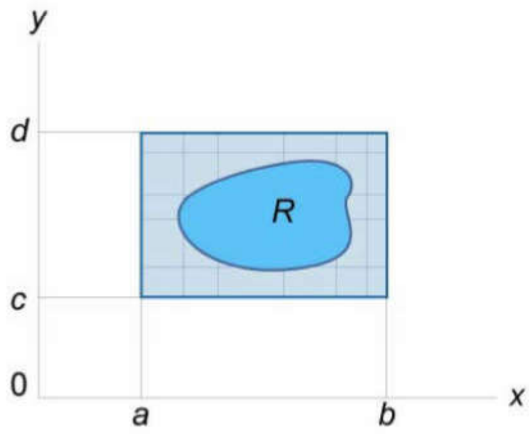

Figure 190.

1079. $\iint_R [f(x,y)+g(x,y)]\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$

1080. $\iint_R [f(x,y)-g(x,y)]\,dA = \iint_R f(x,y)\,dA - \iint_R g(x,y)\,dA$

1081. $\iint_R kf(x,y)\,dA = k\iint_R f(x,y)\,dA$,

where k is a constant.

1082. If $f(x,y)\le g(x,y)$ on R, then $\iint_R f(x,y)\,dA \le \iint_R g(x,y)\,dA$.

1083. If $f(x,y)\ge 0$ on R and $S\subset R$, then

$$\iint_S f(x,y)dA \le \iint_R f(x,y)dA.$$

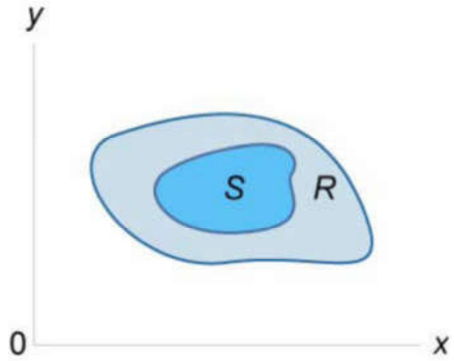

Figure 191.

1084. If $f(x,y) \ge 0$ on R and R and S are non-overlapping regions, then $\iint_{R\cup S} f(x,y)dA = \iint_R f(x,y)dA + \iint_S f(x,y)dA$.

Here $R \cup S$ is the union of the regions R and S.

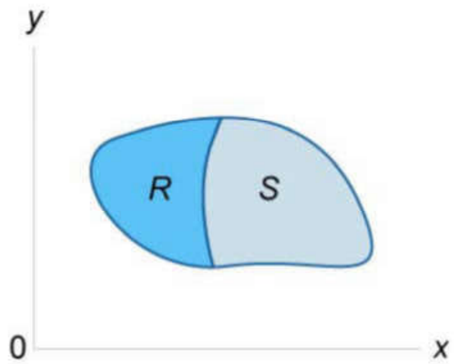

Figure 192.

1085. Iterated Integrals and Fubini's Theorem

$$\iint_R f(x,y)\,dA = \int_a^b \int_{p(x)}^{q(x)} f(x,y)\,dy\,dx$$

for a region of type I,

$R = \{(x,y) \mid a \le x \le b,\ p(x) \le y \le q(x)\}$.

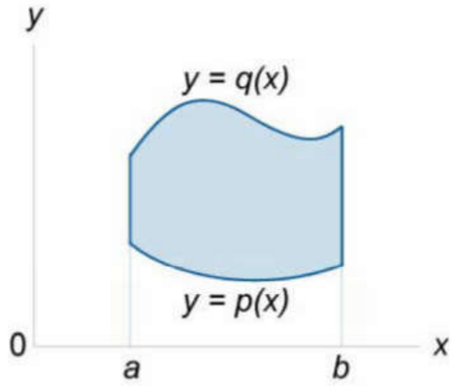

Figure 193.

$$\iint_R f(x,y)\,dA = \int_c^d \int_{u(y)}^{v(y)} f(x,y)\,dx\,dy$$

for a region of type II,

$R = \{(x,y) \mid u(y) \le x \le v(y),\ c \le y \le d\}$.

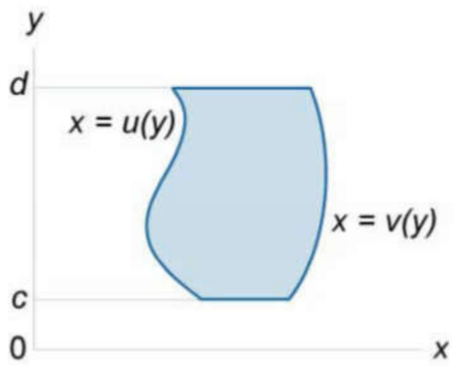

Figure 194.

1086. Double Integrals over Rectangular Regions

If R is the rectangular region $[a,b]\times[c,d]$, then

$$\iint_R f(x,y)dxdy = \int_a^b\left(\int_c^d f(x,y)dy\right)dx = \int_c^d\left(\int_a^b f(x,y)dx\right)dy.$$

In the special case where the integrand $f(x,y)$ can be written as $g(x)h(y)$ we have

$$\iint_R f(x,y)dxdy = \iint_R g(x)h(y)dxdy = \left(\int_a^b g(x)dx\right)\left(\int_c^d h(y)dy\right).$$

1087. Change of Variables

$$\iint_R f(x,y)dxdy = \iint_S f[x(u,v),y(u,v)]\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dudv,$$

where $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0$ is the jacobian of the transformations $(x,y)\to(u,v)$, and S is the pullback of R which can be computed by $x = x(u,v)$, $y = y(u,v)$ into the definition of R.

1088. Polar Coordinates
$x = r\cos\theta$, $y = r\sin\theta$.

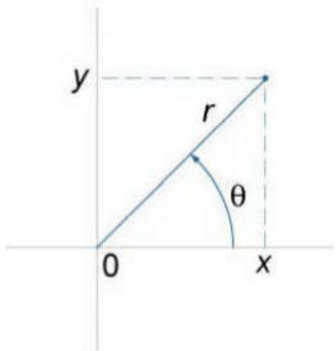

Figure 195.

1089. Double Integrals in Polar Coordinates

The Differential dxdy for Polar Coordinates is
$$dxdy = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| drd\theta = rdrd\theta.$$

Let the region R is determined as follows:
$0 \le g(\theta) \le r \le h(\theta)$, $\alpha \le \theta \le \beta$, where $\beta - \alpha \le 2\pi$.
Then
$$\iint_R f(x,y)dxdy = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta) rdrd\theta.$$

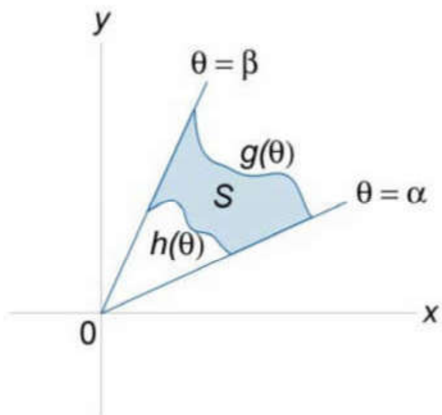

Figure 196.

If the region R is the polar rectangle given by $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$, where $\beta - \alpha \le 2\pi$, then

$$\iint\limits_R f(x,y)\,dxdy = \int\limits_\alpha^\beta \int\limits_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

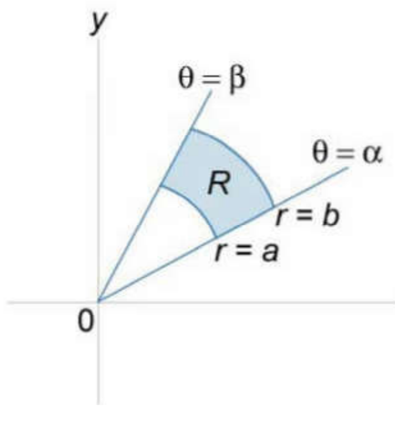

Figure 197.

1090. **Area of a Region**

$$A = \int_a^b \int_{g(x)}^{f(x)} dydx \text{ (for a type I region).}$$

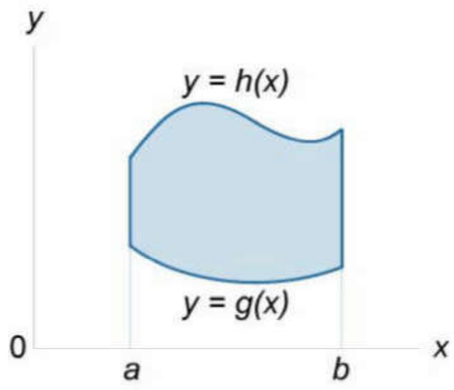

Figure 198.

$$A = \int_c^d \int_{p(y)}^{q(y)} dxdy \text{ (for a type II region).}$$

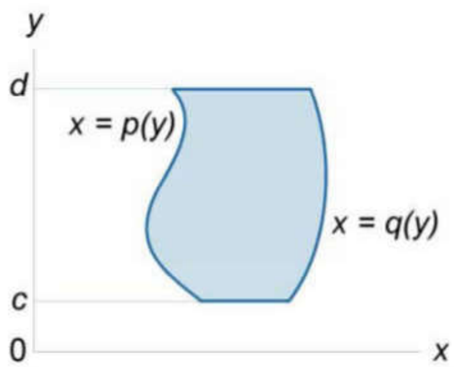

Figure 199.

1091. **Volume of a Solid**

$$V = \iint_R f(x,y)\,dA.$$

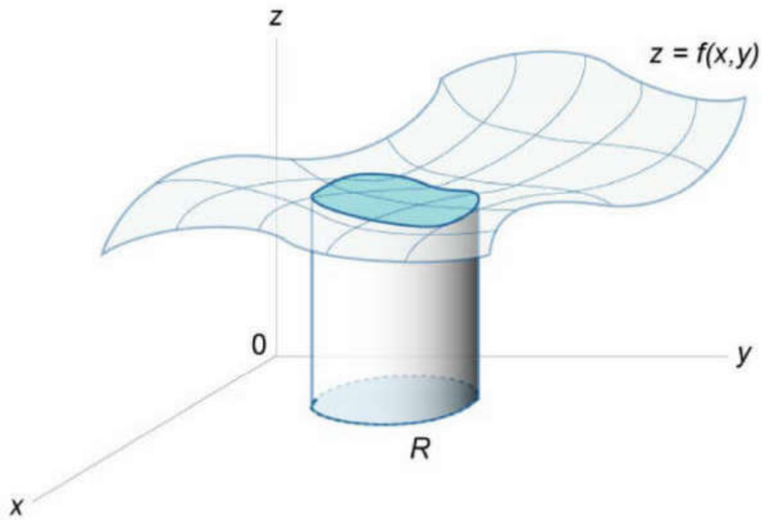

Figure 200.

If R is a type I region bounded by $x = a$, $x = b$, $y = h(x)$, $y = g(x)$, then

$$V = \iint_R f(x,y)\,dA = \int_a^b \int_{h(x)}^{g(x)} f(x,y)\,dy\,dx.$$

If R is a type II region bounded by $y = c$, $y = d$, $x = q(y)$, $x = p(y)$, then

$$V = \iint_R f(x,y)\,dA = \int_c^d \int_{p(y)}^{q(y)} f(x,y)\,dx\,dy.$$

If $f(x,y) \ge g(x,y)$ over a region R, then the volume of the solid between $z_1 = f(x,y)$ and $z_2 = g(x,y)$ over R is given by

$$V = \iint_R [f(x,y) - g(x,y)]dA.$$

1092. Area and Volume in Polar Coordinates

If S is a region in the xy-plane bounded by $\theta = \alpha$, $\theta = \beta$, $r = h(\theta)$, $r = g(\theta)$,

then

$$A = \iint_S dA = \int_\alpha^\beta \int_{h(\theta)}^{g(\theta)} r dr d\theta,$$

$$V = \iint_S f(r,\theta) r dr d\theta.$$

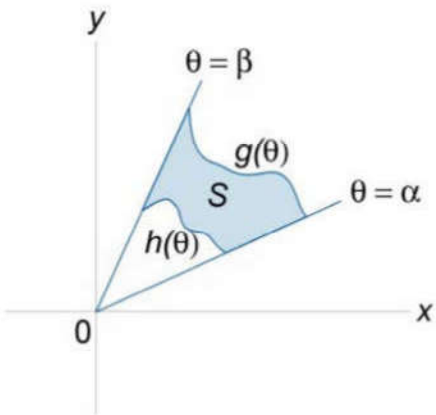

Figure 201.

1093. Surface Area

$$S = \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dxdy$$

1094. Mass of a Lamina

$$m = \iint_R \rho(x, y)\,dA,$$

where the lamina occupies a region R and its density at a point (x,y) is $\rho(x, y)$.

1095. Moments

The moment of the lamina about the x-axis is given by formula

$$M_x = \iint_R y\rho(x, y)\,dA.$$

The moment of the lamina about the y-axis is

$$M_y = \iint_R x\rho(x, y)\,dA.$$

The moment of inertia about the x-axis is

$$I_x = \iint_R y^2\rho(x, y)\,dA.$$

The moment of inertia about the y-axis is

$$I_y = \iint_R x^2\rho(x, y)\,dA.$$

The polar moment of inertia is

$$I_0 = \iint_R \left(x^2 + y^2\right)\rho(x, y)\,dA.$$

1096. Center of Mass

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_R x\rho(x, y)\,dA = \frac{\iint_R x\rho(x, y)\,dA}{\iint_R \rho(x, y)\,dA},$$

$$\bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_R y\rho(x,y)dA = \frac{\iint_R y\rho(x,y)dA}{\iint_R \rho(x,y)dA}.$$

1097. Charge of a Plate

$$Q = \iint_R \sigma(x,y)dA,$$

where electrical charge is distributed over a region R and its charge density at a point (x,y) is $\sigma(x,y)$.

1098. Average of a Function

$$\mu = \frac{1}{S}\iint_R f(x,y)dA,$$

where $S = \iint_R dA$.

9.11 Triple Integral

Functions of three variables: $f(x,y,z)$, $g(x,y,z)$, ...

Triple integrals: $\iiint_G f(x,y,z)dV$, $\iiint_G g(x,y,z)dV$, ...

Riemann sum: $\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} f(u_i, v_j, w_k)\Delta x_i \Delta y_j \Delta z_k$

Small changes: Δx_i, Δy_j, Δz_k

Limits of integration: a, b, c, d, r, s

Regions of integration: G, T, S

Cylindrical coordinates: r, θ, z

Spherical coordinates: r, θ, φ

Volume of a solid: V

Mass of a solid: m
Density: $\mu(x,y,z)$
Coordinates of center of mass: $\bar{x}$, $\bar{y}$, $\bar{z}$
First moments: M_{xy}, M_{yz}, M_{xz}
Moments of inertia: I_{xy}, I_{yz}, I_{xz}, I_x, I_y, I_z, I_0

1099. Definition of Triple Integral
The triple integral over a parallelepiped $[a, b]\times[c, d]\times[r, s]$ is defined to be

$$\iiint\limits_{[a,\,b]\times[c,\,d]\times[r,\,s]} f(x,y,z)dV = \lim_{\substack{\max \Delta x_i \to 0 \\ \max \Delta y_j \to 0 \\ \max \Delta z_k \to 0}} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} f(u_i, v_j, w_k)\Delta x_i \Delta y_j \Delta z_k,$$

where (u_i, v_j, w_k) is some point in the parallelepiped $(x_{i-1}, x_i)\times(y_{j-1}, y_j)\times(z_{k-1}, z_k)$, and $\Delta x_i = x_i - x_{i-1}$, $\Delta y_j = y_j - y_{j-1}$, $\Delta z_k = z_k - z_{k-1}$.

1100. $$\iiint\limits_G [f(x,y,z)+g(x,y,z)]dV = \iiint\limits_G f(x,y,z)dV + \iiint\limits_G g(x,y,z)dV$$

1101. $$\iiint\limits_G [f(x,y,z)-g(x,y,z)]dV = \iiint\limits_G f(x,y,z)dV - \iiint\limits_G g(x,y,z)dV$$

1102. $$\iiint\limits_G kf(x,y,z)dV = k\iiint\limits_G f(x,y,z)dV,$$
where k is a constant.

1103. If $f(x,y,z)\ge 0$ and G and T are nonoverlapping basic regions, then
$$\iiint\limits_{G\cup T} f(x,y,z)dV = \iiint\limits_G f(x,y,z)dV + \iiint\limits_T f(x,y,z)dV.$$
Here $G\cup T$ is the union of the regions G and T.

1104. Evaluation of Triple Integrals by Repeated Integrals

If the solid G is the set of points (x,y,z) such that $(x,y)\in R$, $\chi_1(x,y)\le z\le \chi_2(x,y)$, then

$$\iiint_G f(x,y,z)\,dxdydz=\iint_R\left[\int_{\chi_1(x,y)}^{\chi_2(x,y)} f(x,y,z)\,dz\right]dxdy,$$

where R is projection of G onto the xy-plane.

If the solid G is the set of points (x,y,z) such that $a\le x\le b$, $\varphi_1(x)\le y\le \varphi_2(x)$, $\chi_1(x,y)\le z\le \chi_2(x,y)$, then

$$\iiint_G f(x,y,z)\,dxdydz=\int_a^b\left[\int_{\varphi_1(x)}^{\varphi_2(x)}\left(\int_{\chi_1(x,y)}^{\chi_2(x,y)} f(x,y,z)\,dz\right)dy\right]dx$$

1105. Triple Integrals over Parallelepiped

If G is a parallelepiped $[a,\,b]\times[c,\,d]\times[r,\,s]$, then

$$\iiint_G f(x,y,z)\,dxdydz=\int_a^b\left[\int_c^d\left(\int_r^s f(x,y,z)\,dz\right)dy\right]dx.$$

In the special case where the integrand $f(x,y,z)$ can be written as $g(x)h(y)k(z)$ we have

$$\iiint_G f(x,y,z)\,dxdydz=\left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right)\left(\int_r^s k(z)\,dz\right).$$

1106. Change of Variables

$$\iiint_G f(x,y,z)\,dxdydz=$$

$$=\iiint_S f[x(u,v,w),y(u,v,w),z(u,v,w)]\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|dxdydz,$$

where $\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \neq 0$ is the jacobian of the transformations $(x,y,z) \to (u,v,w)$, and S is the pull-back of G which can be computed by $x = x(u,v,w)$,
$y = y(u,v,w)$
$z = z(u,v,w)$ into the definition of G.

1107. Triple Integrals in Cylindrical Coordinates
The differential dxdydz for cylindrical coordinates is

$$dxdydz = \left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| drd\theta dz = rdrd\theta dz.$$

Let the solid G is determined as follows:
$(x,y) \in R,\ \chi_1(x,y) \le z \le \chi_2(x,y)$,
where R is projection of G onto the xy-plane. Then

$$\iiint_G f(x,y,z)dxdydz = \iiint_S f(r\cos\theta, r\sin\theta, z)rdrd\theta dz$$

$$= \iint_{R(r,\theta)} \left[\int_{\chi_1(r\cos\theta, r\sin\theta)}^{\chi_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)dz \right] rdrd\theta.$$

Here S is the pullback of G in cylindrical coordinates.

1108. Triple Integrals in Spherical Coordinates
The Differential dxdydz for Spherical Coordinates is

$$dxdydz = \left|\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}\right| drd\theta d\varphi = r^2 \sin\theta drd\theta d\varphi$$

$$\iiint_G f(x,y,z)dxdydz =$$